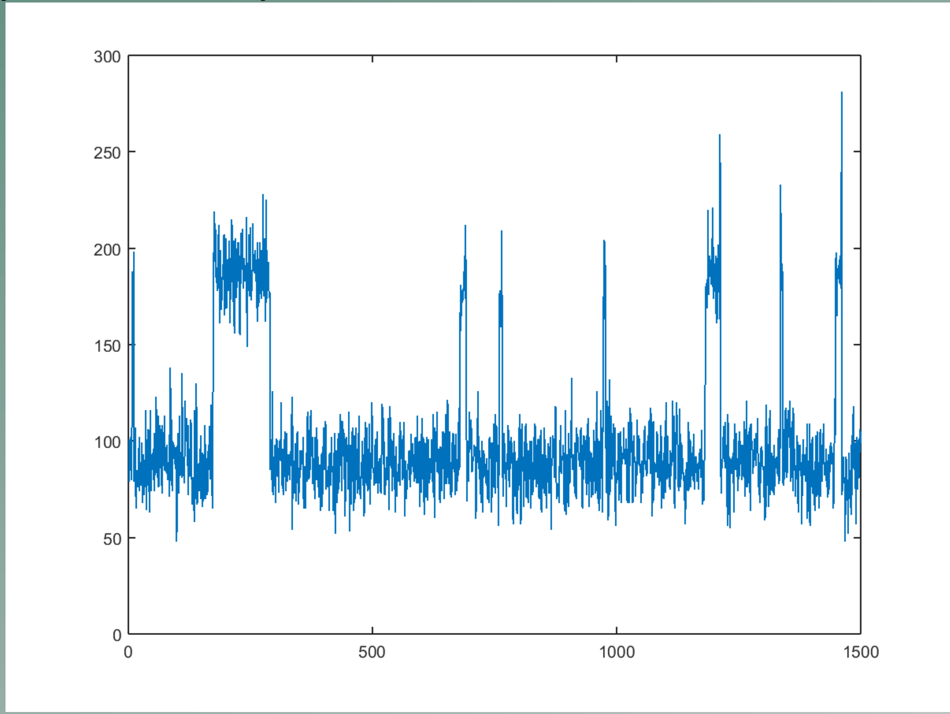
Wavelets

Using Multiresolution Analysis for edge detection

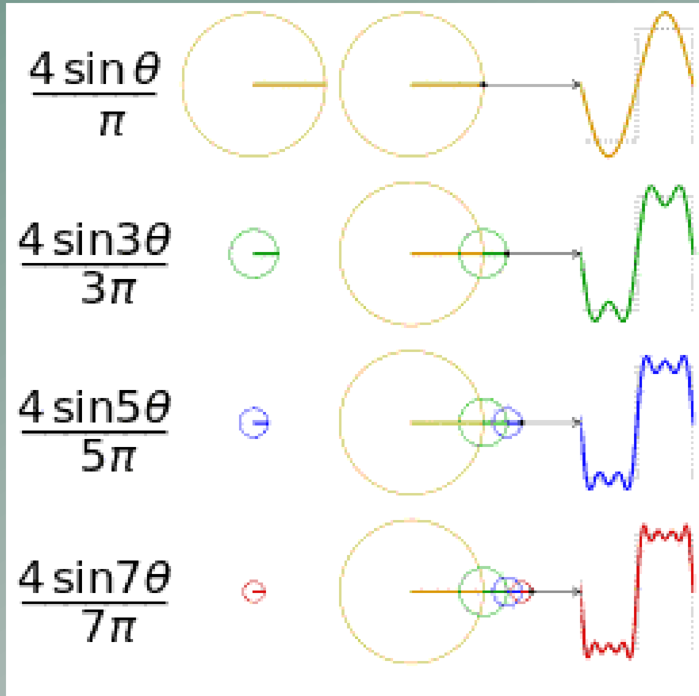
What’s the problem

- Both 2D grayscale images and 1D digital signals are subject to sudden changes over the domain of interest
  - Images – Random changes image to image (machine vision, edge detection)
  - Signal – Random in time domain (RTS noise)

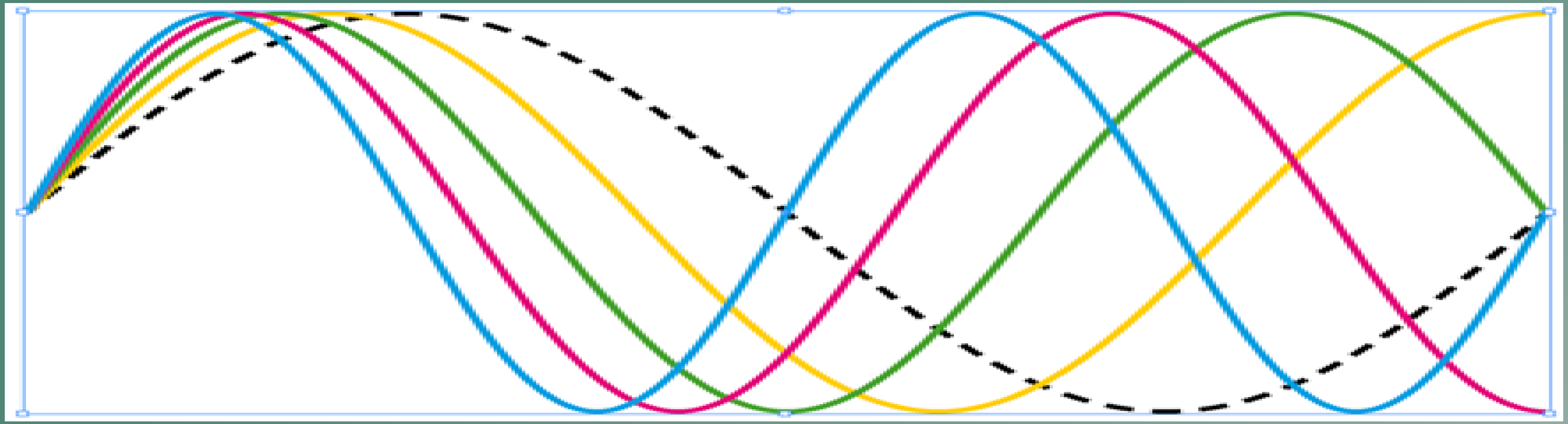


Fourier Analysis

- Most common approach to signal analysis
- Struggles with random and discrete changes
  - Uses unbounded sine waves to construct a signal
  - Inherently periodic, and therefore have difficulty reproducing non-periodic features.
- Only returns information about the collective frequency density of a signal. All time based data is disregarded.

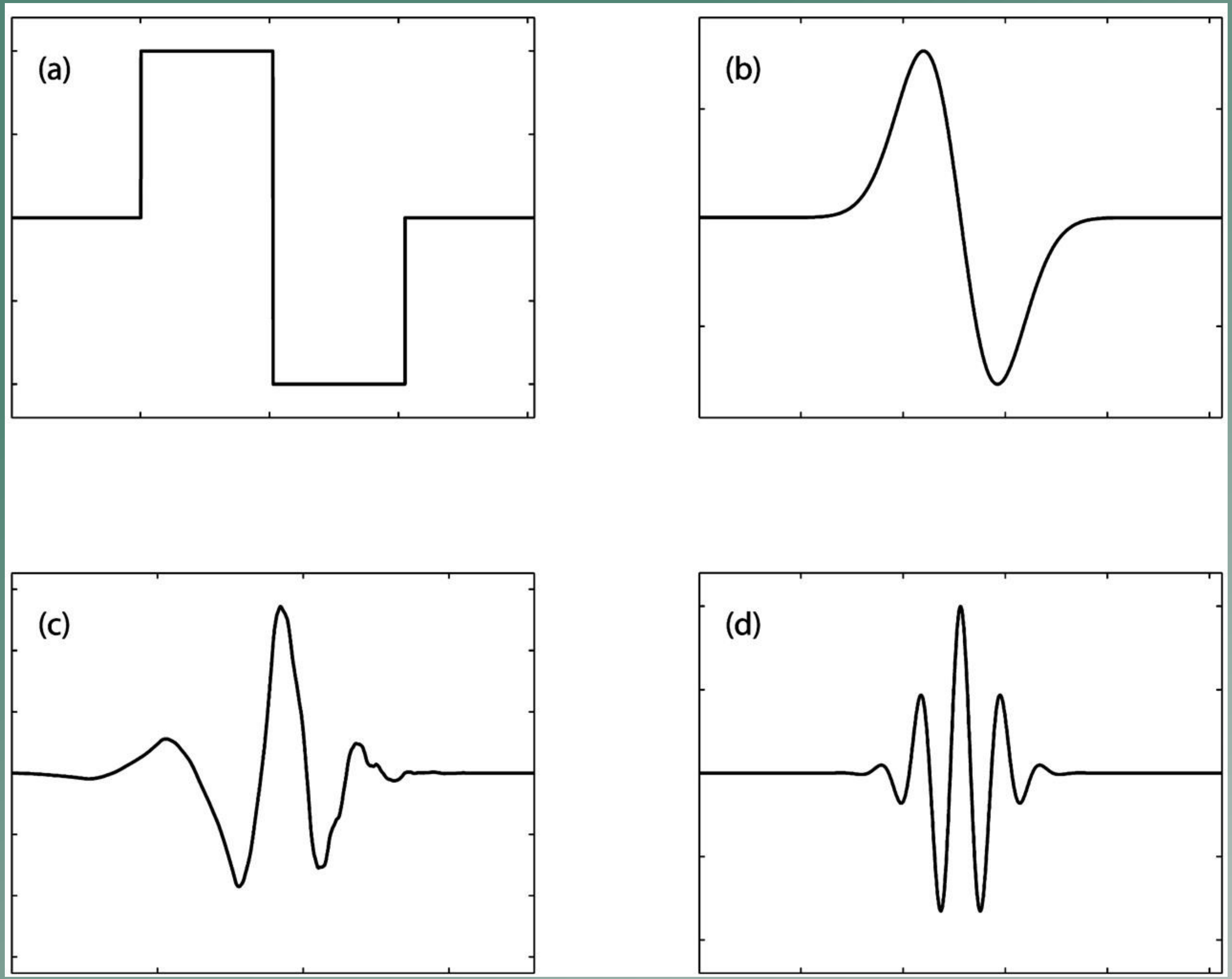


Fourier Analysis cont.



- Since time domain data is important to many physical phenomena, Fourier analysis may be insufficient.
- What would be nice is a method that uses a basis set of functions that are bounded and can tell someone when an event occurs.
- A basis set made out of…

Wavelets

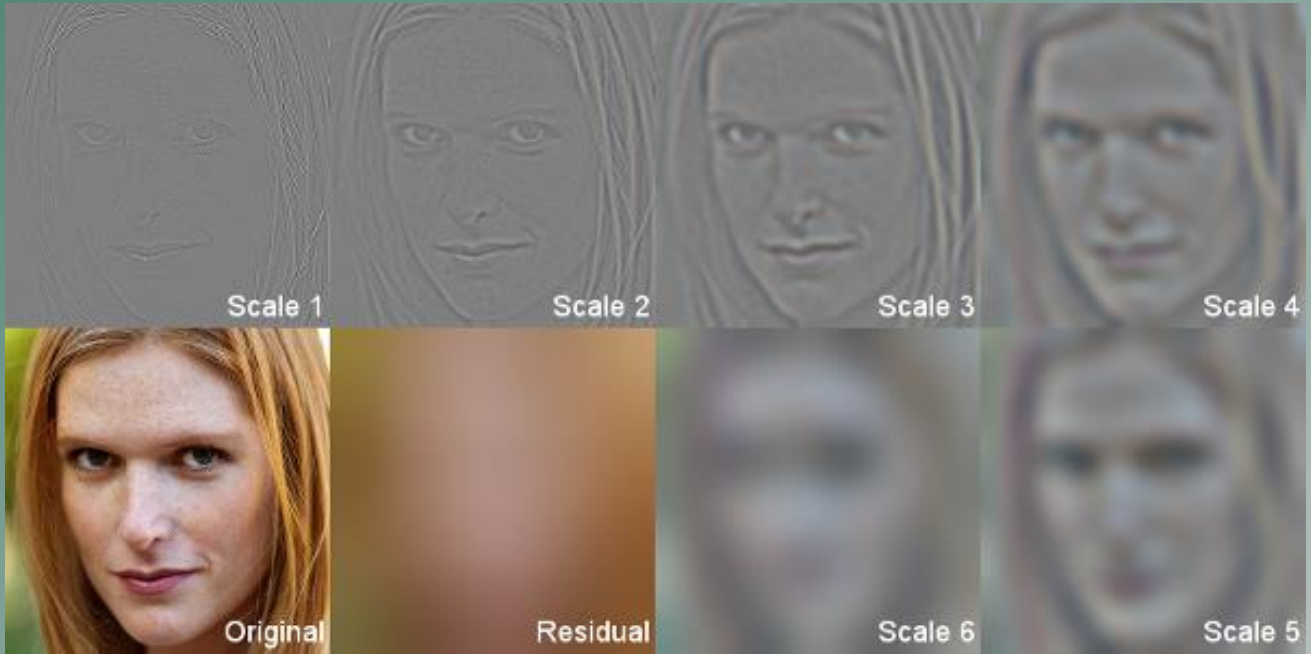


Wavelet Transform

- There are two kinds
  - Continuous wavelet transform
  - Discrete wavelet transform
- Continuous
  - Useful for phenomena like wind speed or earthquake tremors.
- Discrete
  - Useful for sudden and random changes like RTS noise.

Wavelet Transform

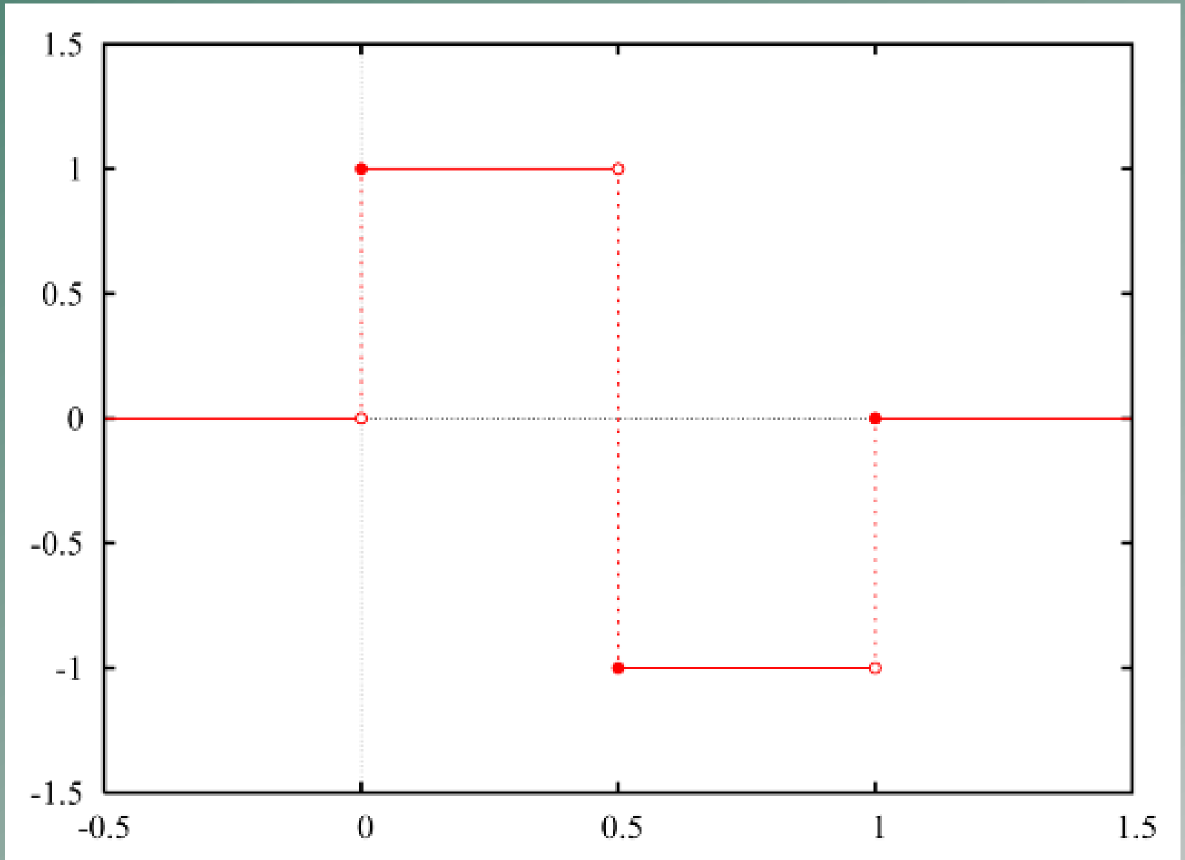
- In essence: The study of the cracks between resolutions
  - Start with a blurry image (squinty eyes)
  - Slowly focus
  - Wavelets (MRA) deal with the difference, or what makes an improvement between these levels.



The Haar Wavelet cont.

- To construct the h(t) signal we can use the Haar wavelet as a template of the representation to be dilated and translated.

$$\psi(t) = \begin{cases} 1, & 0 \le t < 0.5 \\ -1, & 0.5 \le t < 1 \end{cases}$$



The Haar Wavelet cont.

- The general form:

$$\psi(\frac{t-\tau}{S})$$

τ: Translation index

- Moves wavelet across the signal

S: Dilation index

- Larger S, faster freq. info provided.
- Smaller S, lower freq. info provided.

Every time we translate and dilate we gain new information about the signal.

Haar Wavelet Operator

$$\mathbf{W}_1^1 = \frac{1}{\sqrt{2}}(1, -1, 0, 0, 0, 0, 0, 0)$$

$$\mathbf{W}_2^1 = \frac{1}{\sqrt{2}}(0, 0, 1, -1, 0, 0, 0, 0)$$

Haar Wavelet Operator

$$d_1 = \frac{f_1 - f_2}{\sqrt{2}} = \mathbf{f} \cdot \mathbf{W_1^1}$$

$$d_m = \mathbf{f} \cdot \mathbf{W_m^1}$$

Haar Scaling Operator

$$\mathbf{V}_1^1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0, 0, 0, 0, 0)$$

$$\mathbf{V}_2^1 = \frac{1}{\sqrt{2}}(0, 0, 1, 1, 0, 0, 0, 0)$$

Haar Scaling Operator cont.

$$a_1 = \frac{f_1 + f_2}{\sqrt{2}} = \mathbf{f} \cdot \mathbf{V_1^1}$$

$$a_m = \mathbf{f} \cdot \mathbf{V_m^1}$$

Haar Wavelets

- Consider a discrete signal of length N

$$\mathbf{f} = (f_1, f_2, f_3, \ldots f_m)$$

  - Haar wavelet analysis creates two daughter signals out of the mother signal

    - One signal is called the trend signal. It is made of coefficients such that

$$a_1 = \frac{f_1 + f_2}{\sqrt{2}} \qquad a_2 = \frac{f_3 + f_4}{\sqrt{2}}$$

Haar Wavelets

- The general form of the trend signal is:

$$a_m = \frac{f_{2m-1} + f_{2m}}{\sqrt{2}}$$

For m up to N/2

Haar Wavelets

- The other daughter signal contains the details.

$$d_1 = \frac{f_1 - f_2}{\sqrt{2}} \quad d_2 = \frac{f_3 - f_4}{\sqrt{2}}$$

Haar Wavelets

- The general form of the details signal is:

$$d_m = \frac{f_{2m-1} - f_{2m}}{\sqrt{2}}$$

For m up to N/2

Haar Wavelets – Example (First Level)

$$\mathbf{f} = (4, 6, 10, 12, 8, 6, 5, 5)$$

$$\mathbf{a^1} = (5\sqrt{2}, 11\sqrt{2}, 7\sqrt{2}, 5\sqrt{2})$$

$$\mathbf{d^1} = (-\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$$

Haar Wavelets – Inverse Transform

$$\mathbf{f} = \left(\frac{a_1 + d_1}{\sqrt{2}}, \frac{a_1 - d_1}{\sqrt{2}}, \ldots, \frac{a_{\frac{N}{2}} + d_{\frac{N}{2}}}{\sqrt{2}}, \frac{a_{\frac{N}{2}} - d_{\frac{N}{2}}}{\sqrt{2}}\right)$$

Haar Wavelet – Cons. Of Energy

- A way to consider the 'energy' of a signal is by taking the sum of the squares (just like how we calculate total kinetic energy)

$$\varepsilon_{\mathbf{f}} = (f_1^2 + f_2^2 + f_3^2 + \ldots + f_N^2 +)$$

$$\varepsilon_{\mathbf{f}} = (4^2 + 6^2 + \ldots + 5^2 +) = 446$$

Haar Wavelet- Cons. Of Energy

$$\mathbf{a^1} = (5\sqrt{2}, 11\sqrt{2}, 7\sqrt{2}, 5\sqrt{2})$$

$$\varepsilon_{\mathbf{a_1}} = (25*2+121*2+49*2+25*2) = 440$$

Haar Wavelet- Cons. Of Energy

$$\mathbf{d}^{\mathbf{1}} = (-\sqrt{2}, -\sqrt{2}, \sqrt{2}, 0)$$

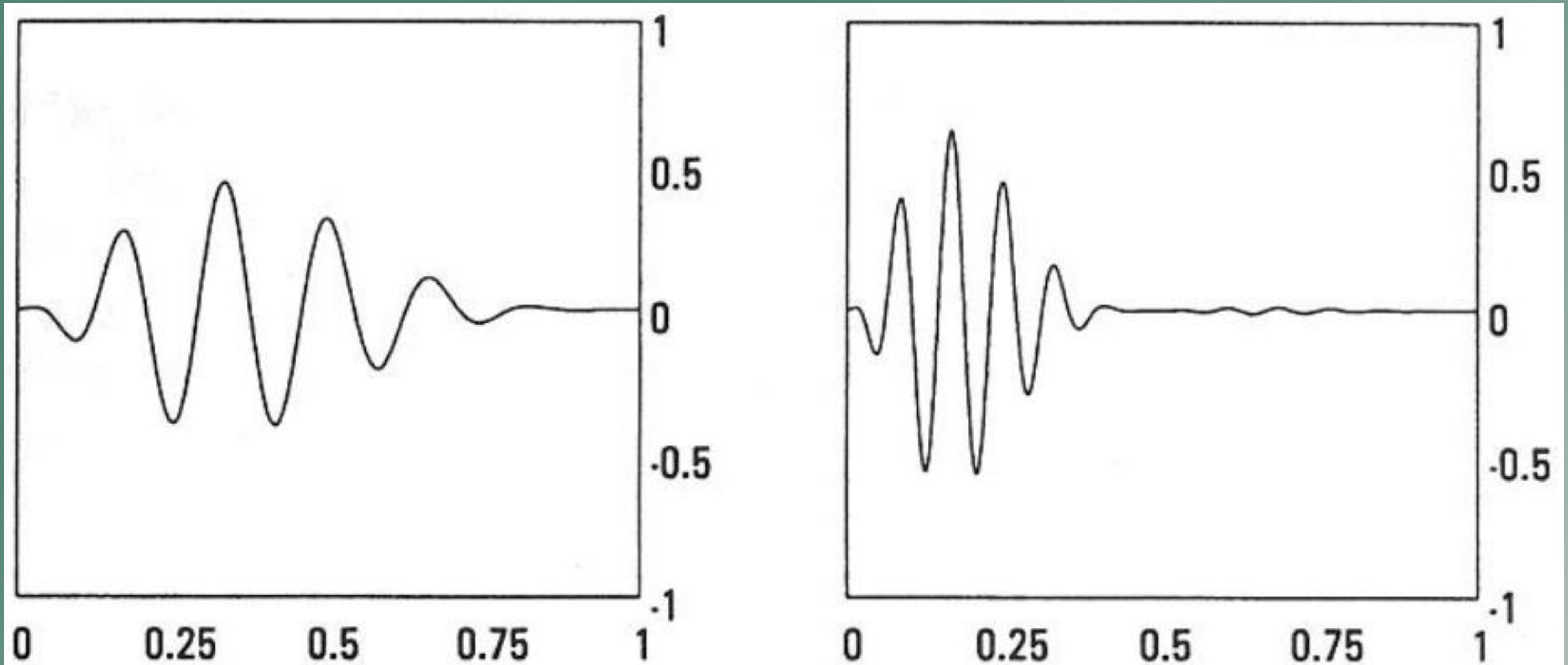
$$\varepsilon_{\mathbf{d_1}} = (2 + 2 + 2 + 0) = 6$$

Haar Wavelet- Cons. Of Energy

- Since $E_f = E_{a1}+E_{d1}$ the ‘energy’ of the signal is preserved. More over, the vast majority of available ‘energy’ ends up in the trends daughter signal.

$$\frac{440}{446} = 98.7\%$$

Haar Wavelet- Cons. Of Energy

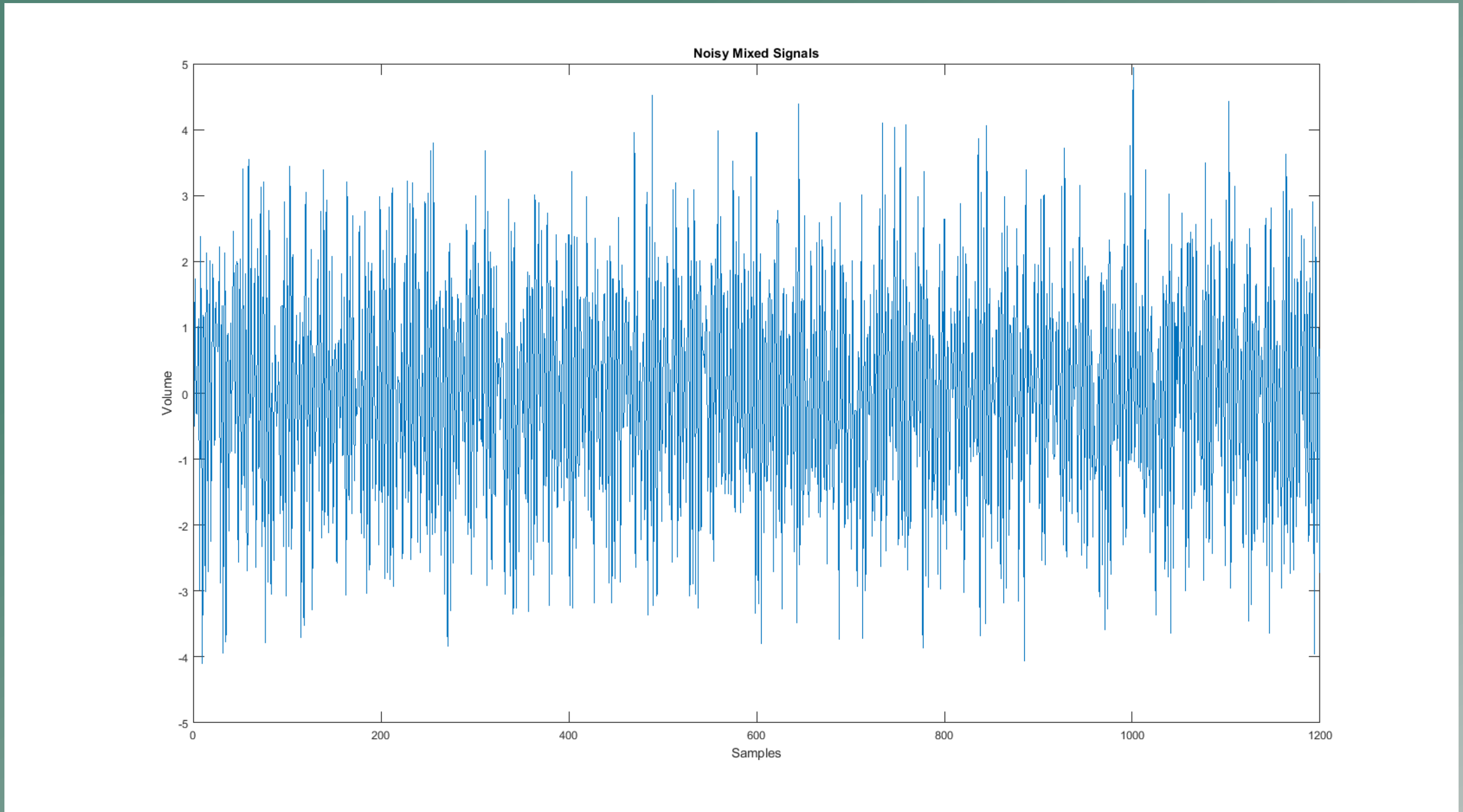


Analysis Performance Comparison

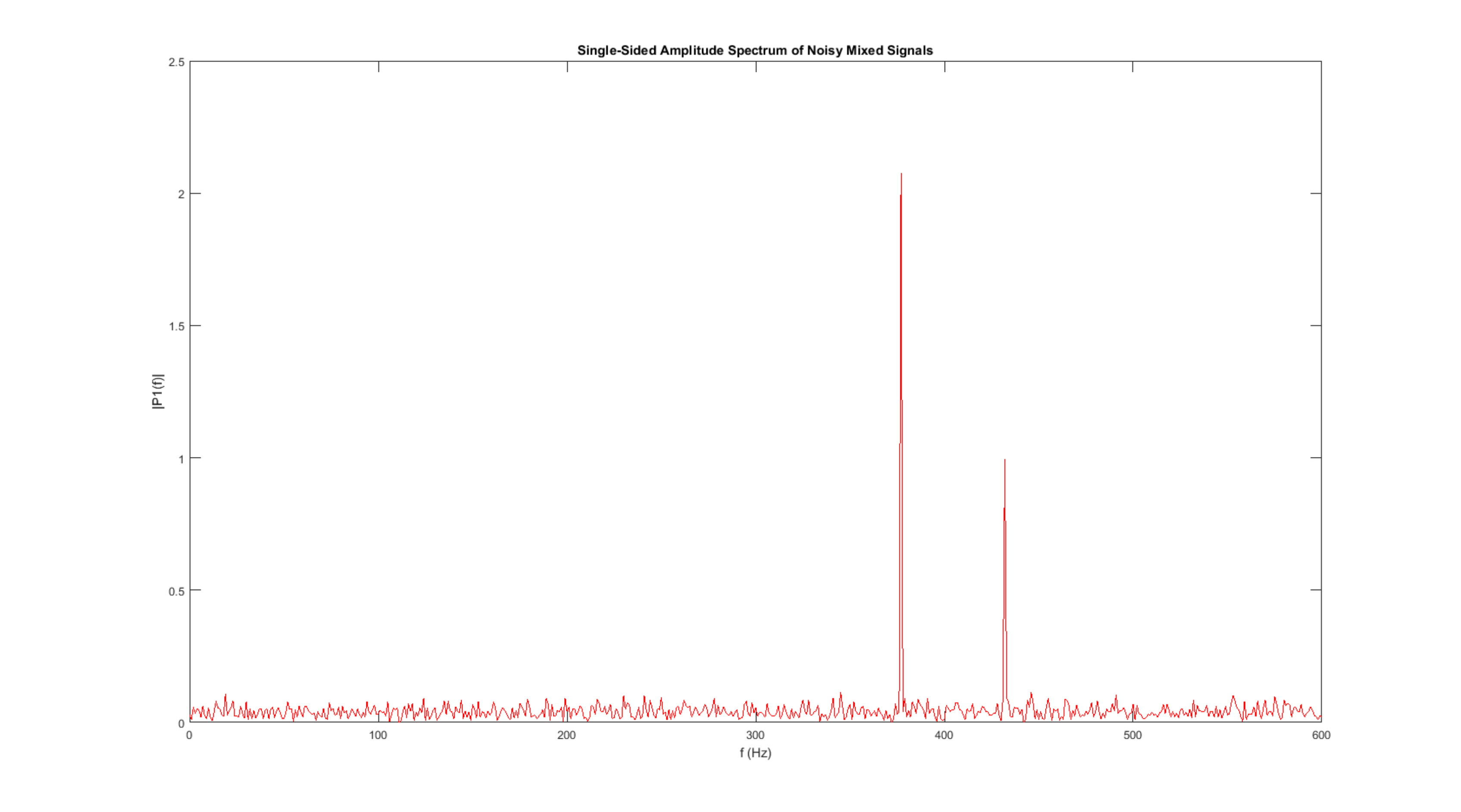
- 3 kinds of signals
  - Noisy Mixed Signals
  - Music Signals
  - Noisy Pixel Output

- 3 kinds of analysis
  - Fourier Transform
  - Short Time Fourier Transform
  - Wavelet Transform

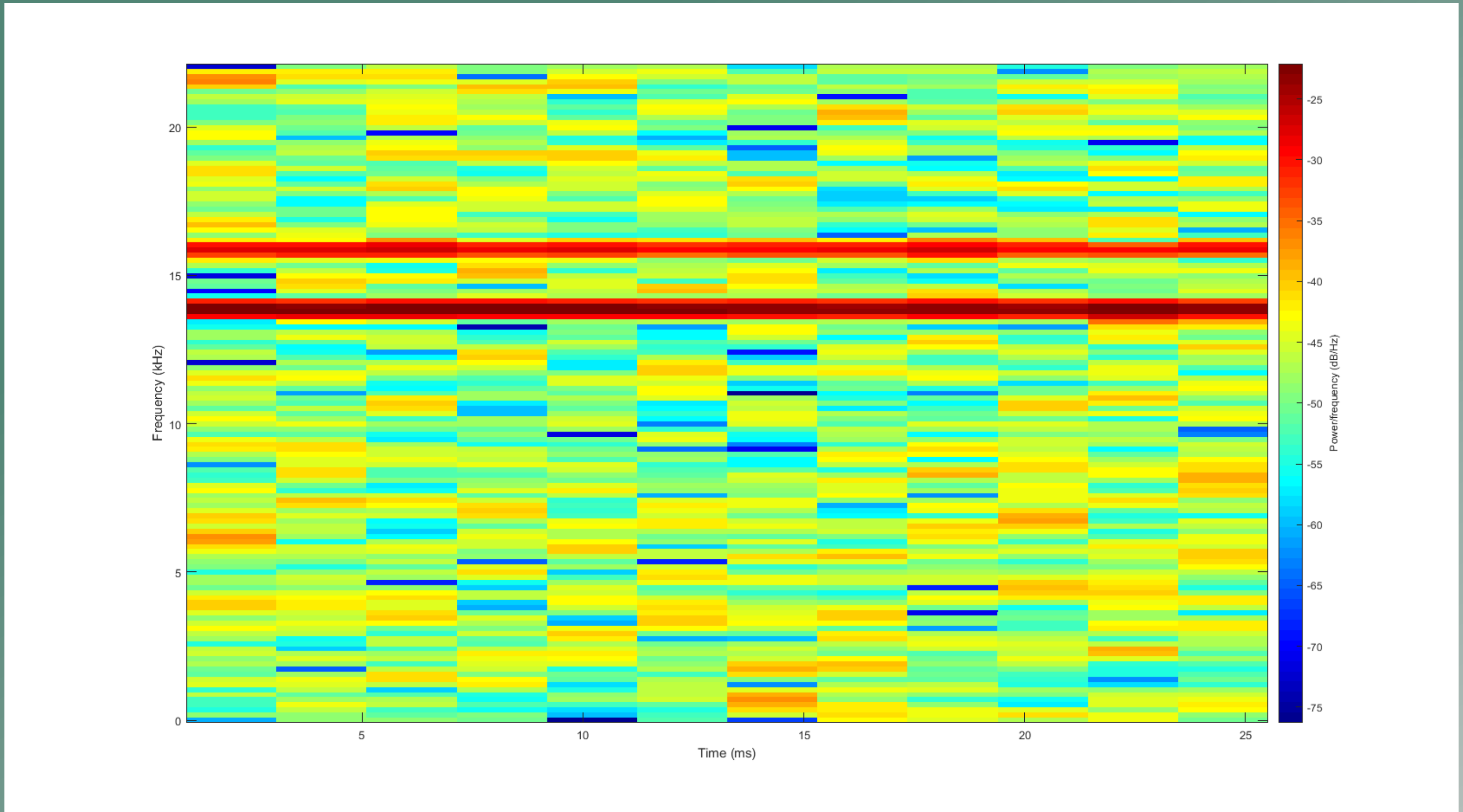
Mixed Signal-Time



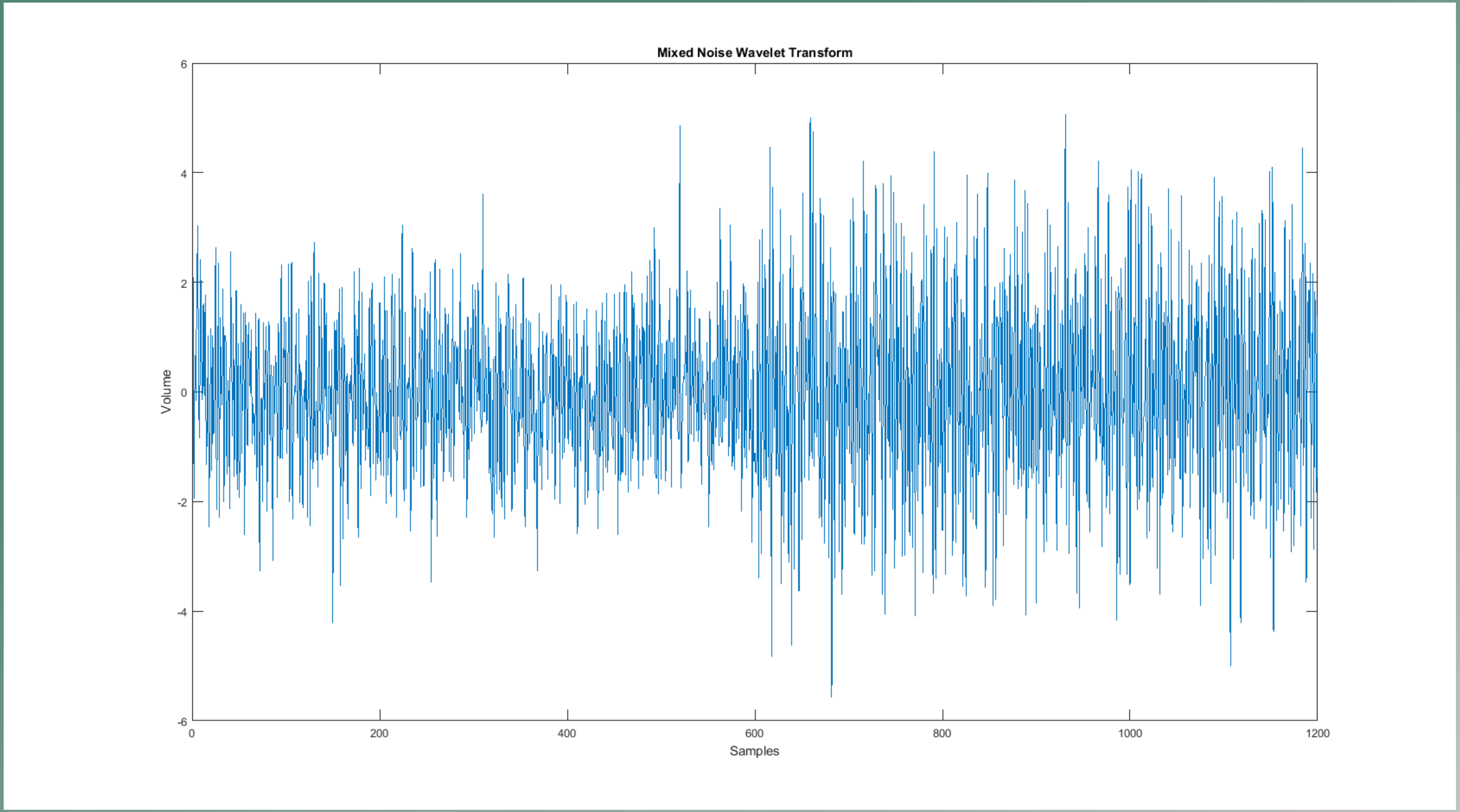
Mixed Signal-Fourier Transform



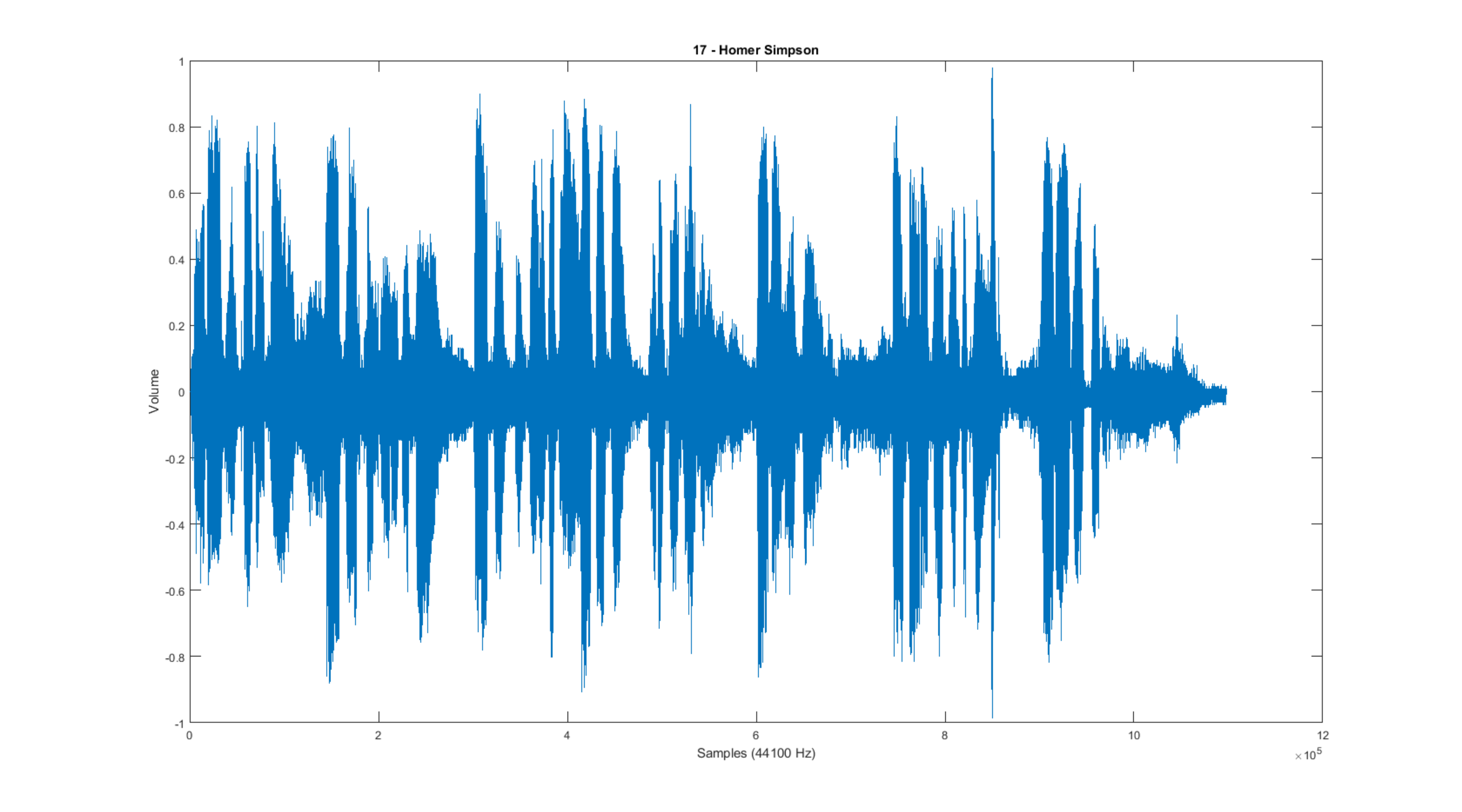
Mixed Signal-STFT



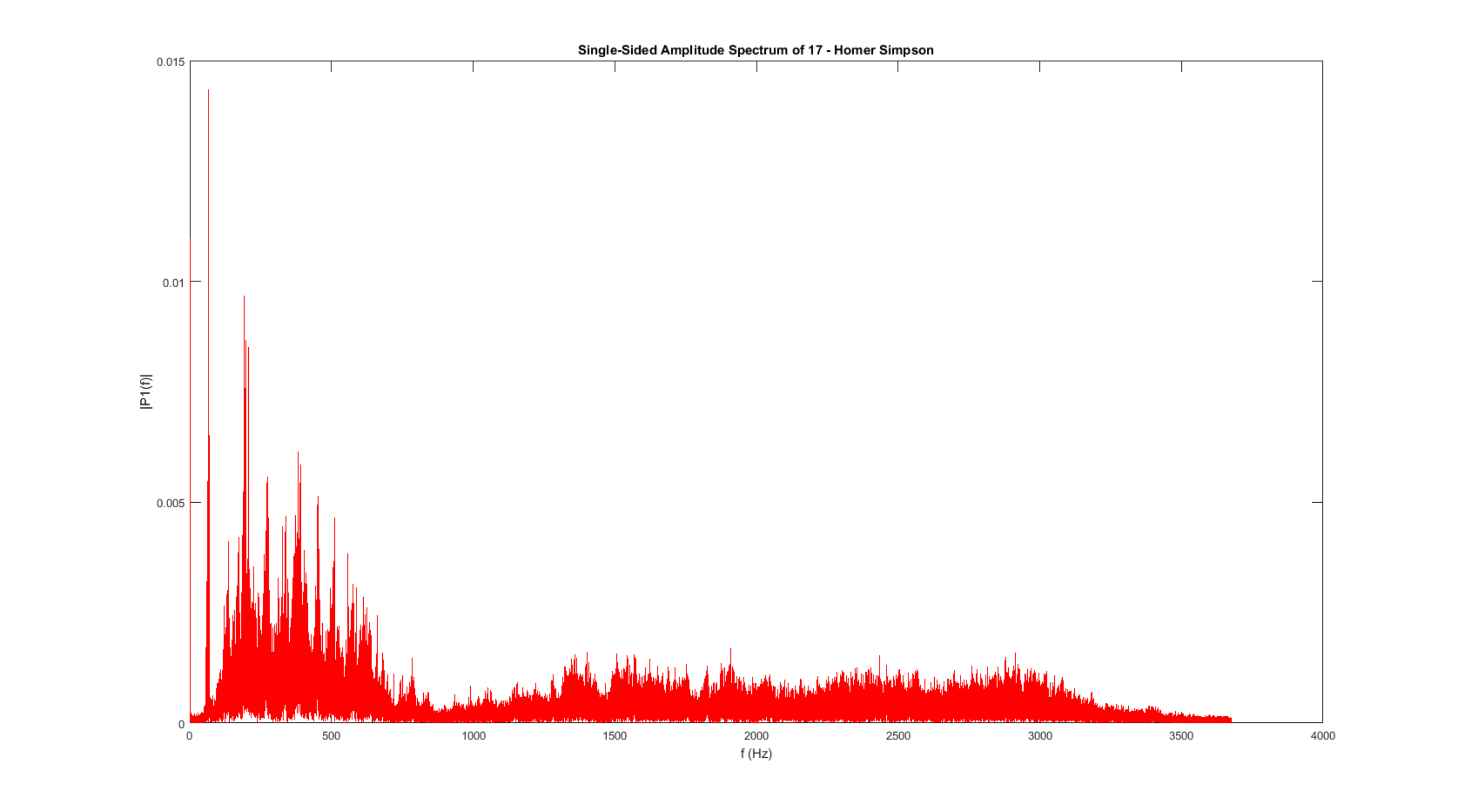
Mixed Signal – Wavelet Transform



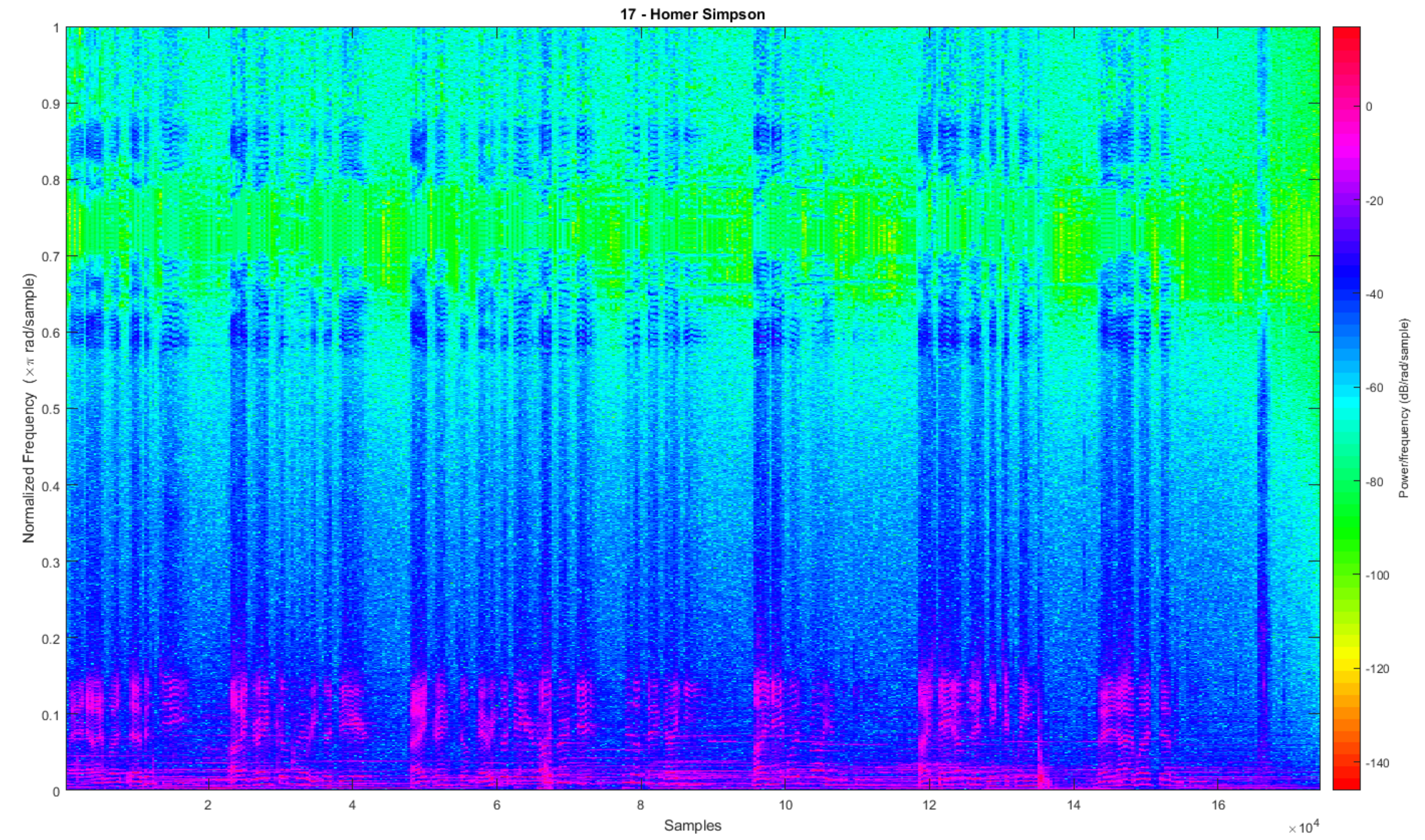
Sound Waves-Time (link to .wave file)



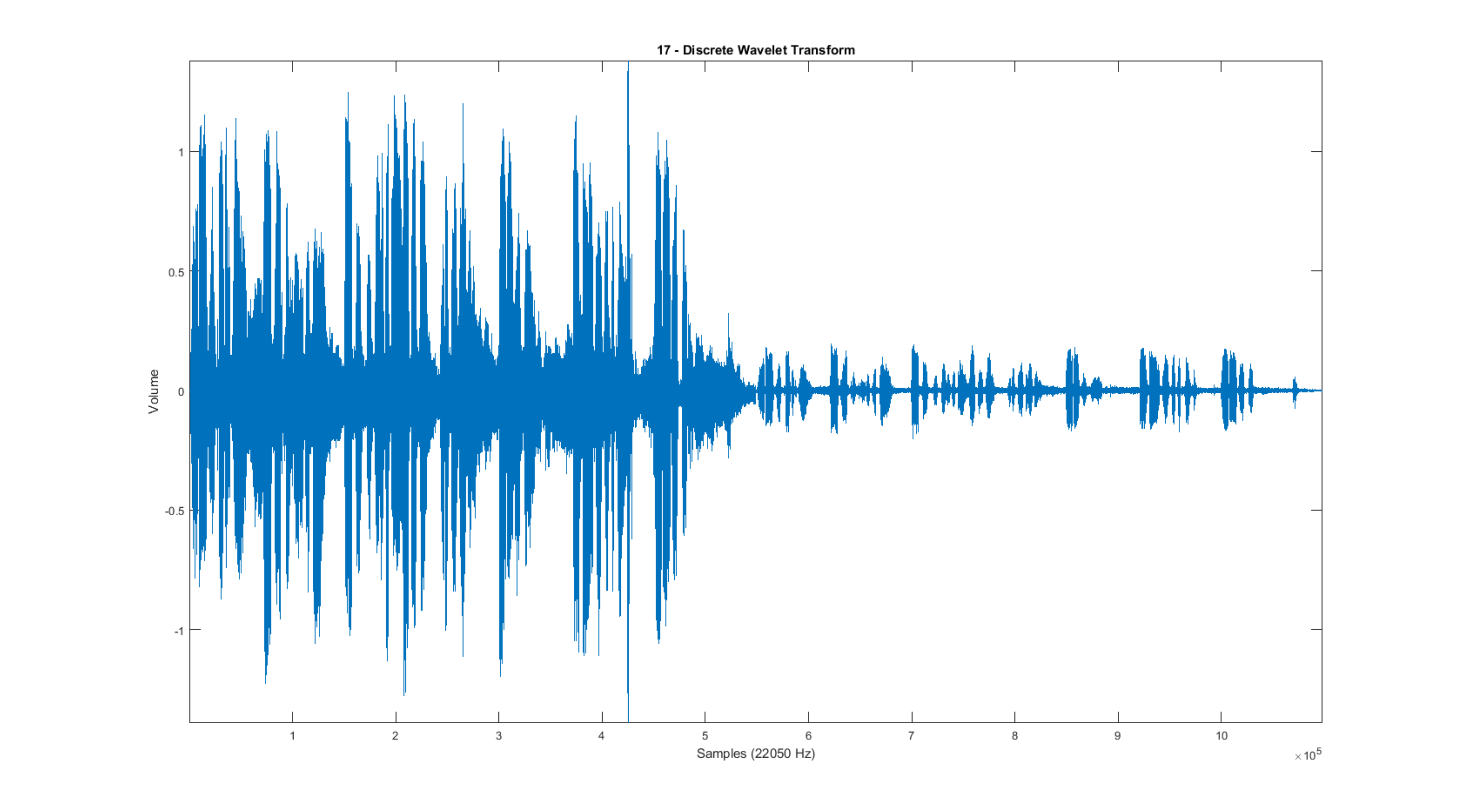
Sound Wave-Fourier Transform



Sound Wave- STFT



Sound Wave – Wavelet Transform

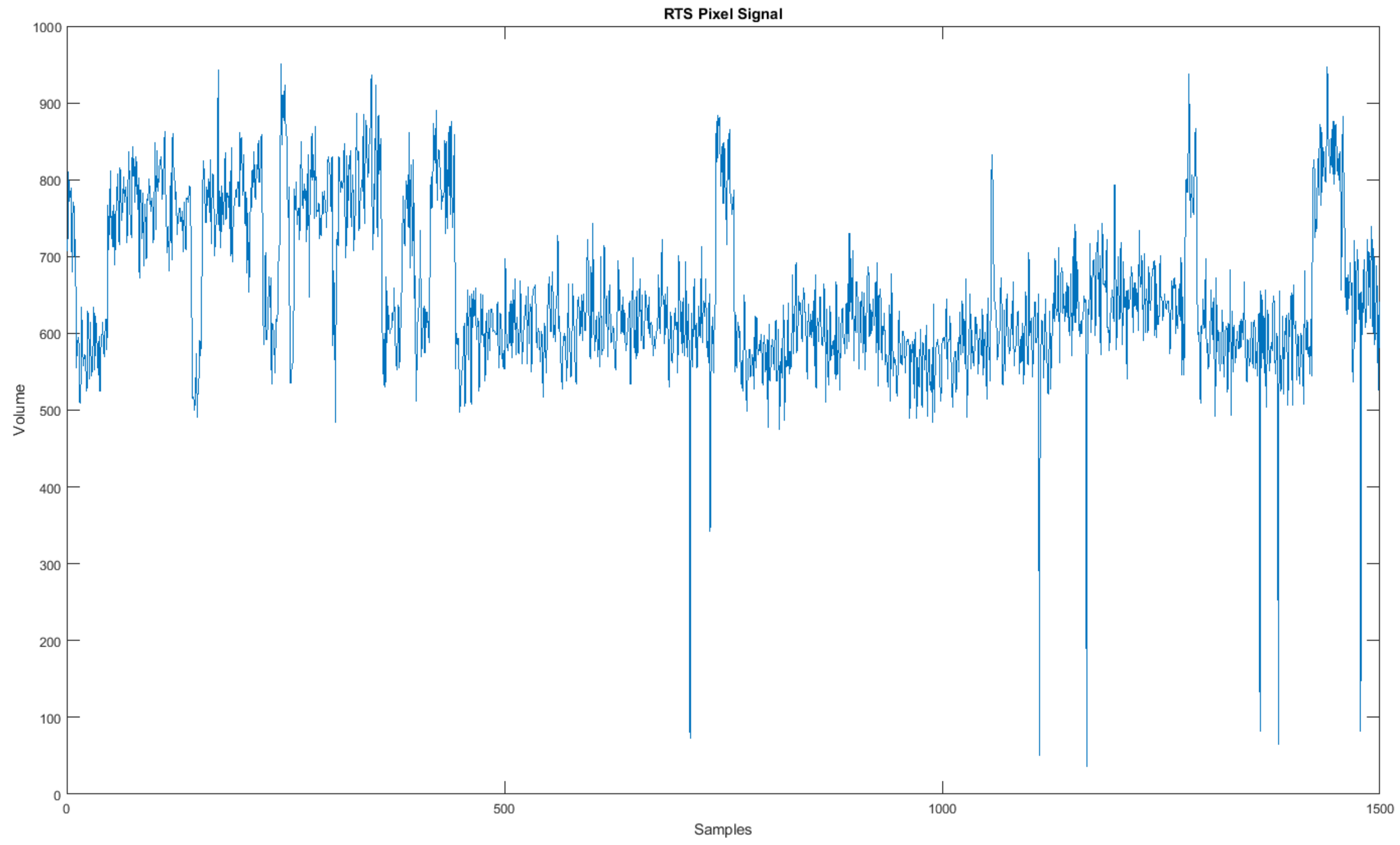


Sound Wave – Compression

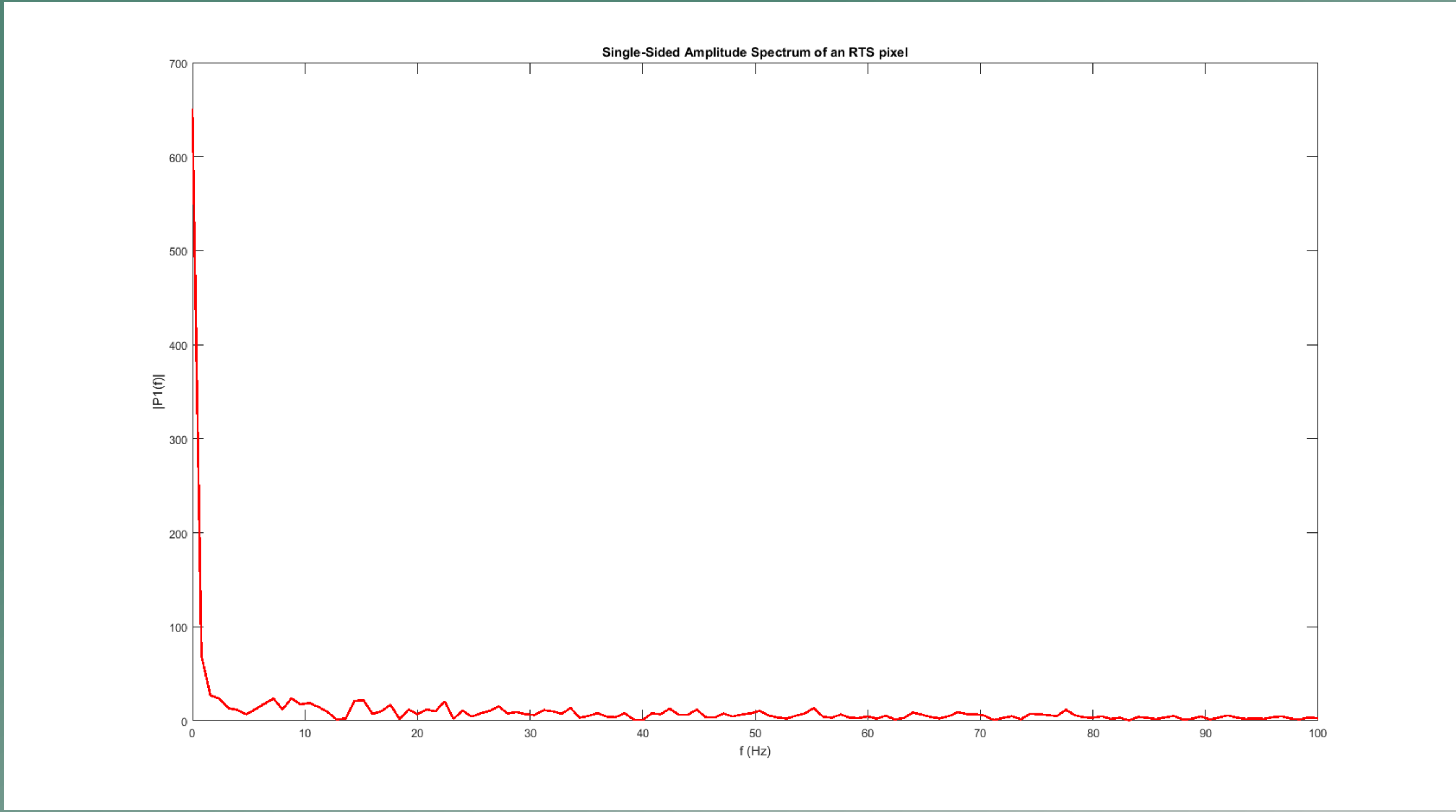
Original

Compressed

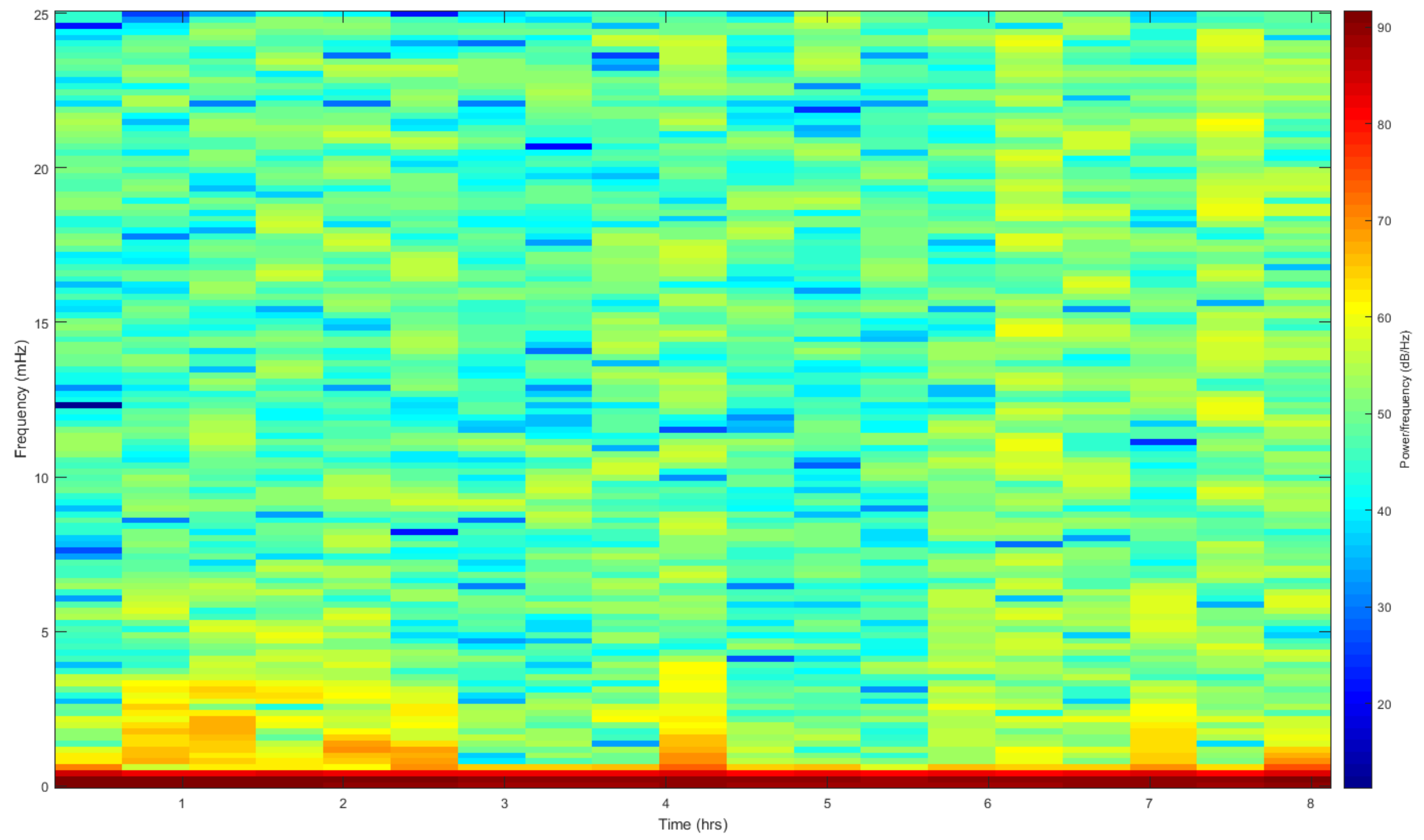
Pixel Output - Time



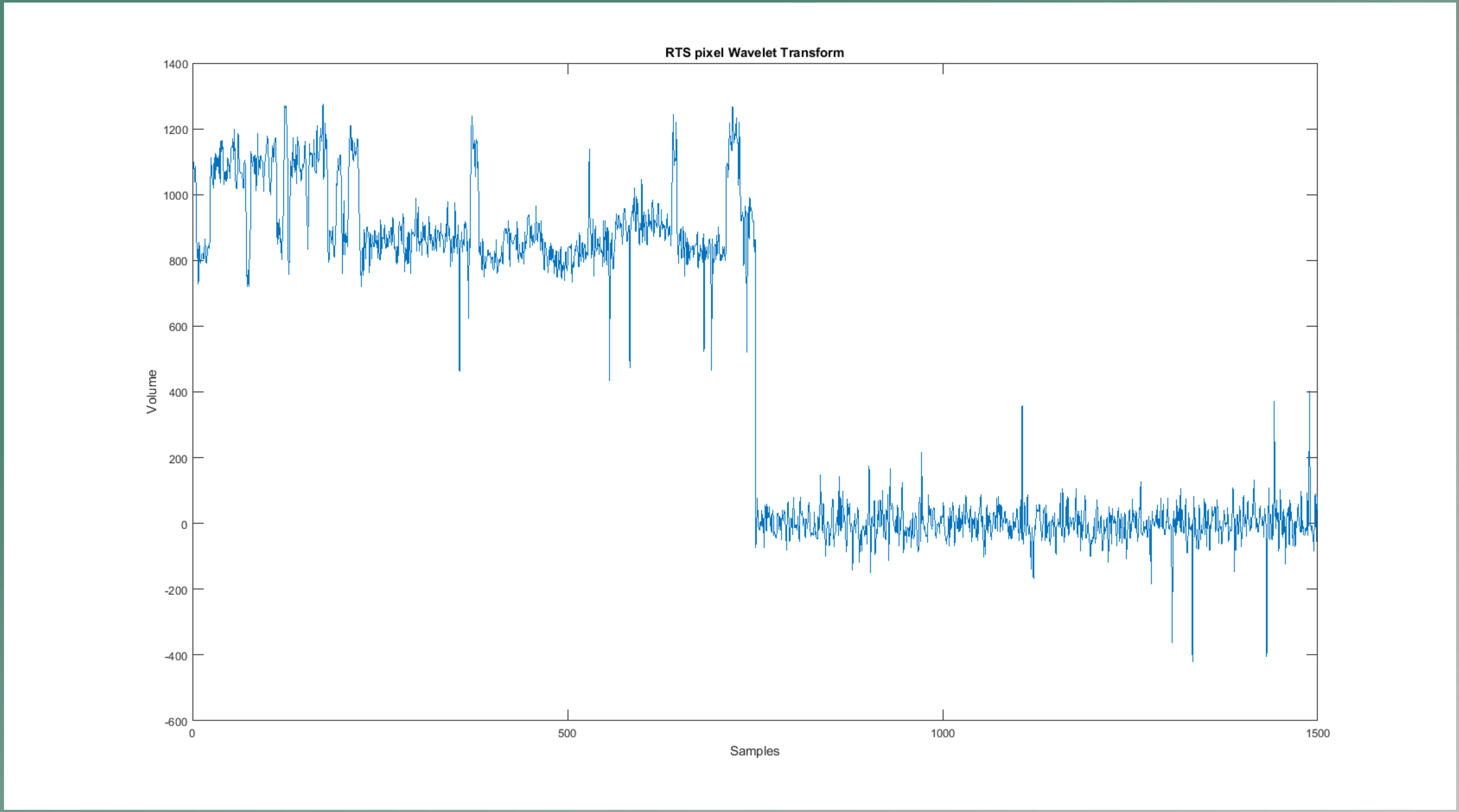
Pixel Output – Fourier Transform



Pixel Output - STFT



Pixel Output – Wavelet Transform

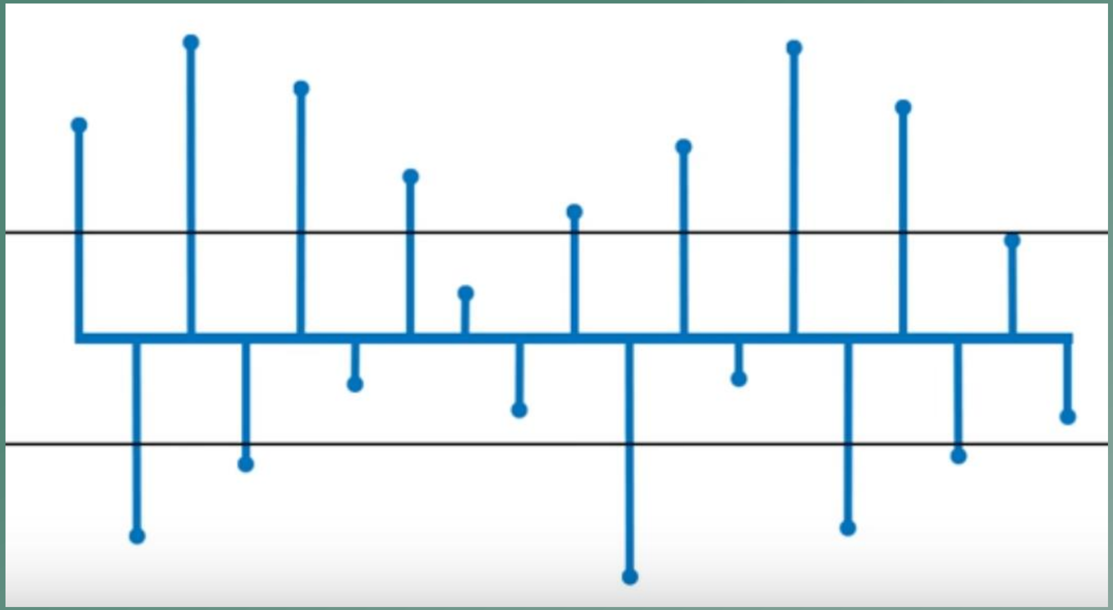


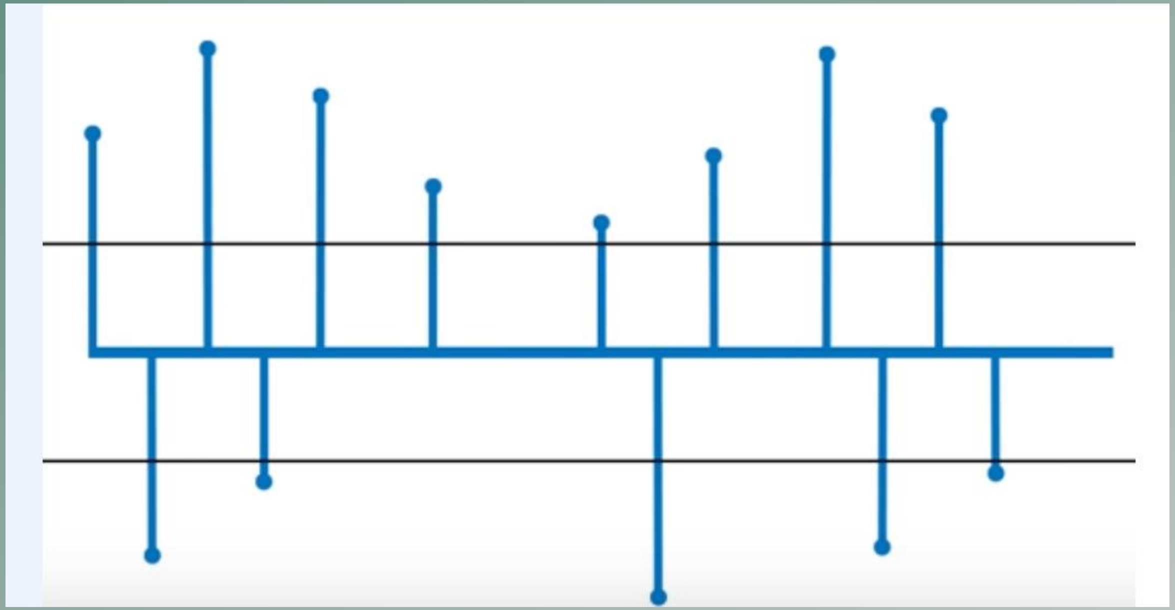
Wavelet Denoising

- Trend coefficients are useful in compression
- Details coefficients are useful for denoising
  - Removes white noise
  - Retains large scale structure

Wavelet Denoising

- Thresholds
  - Set a Threshold Value (Derived from Statistics)





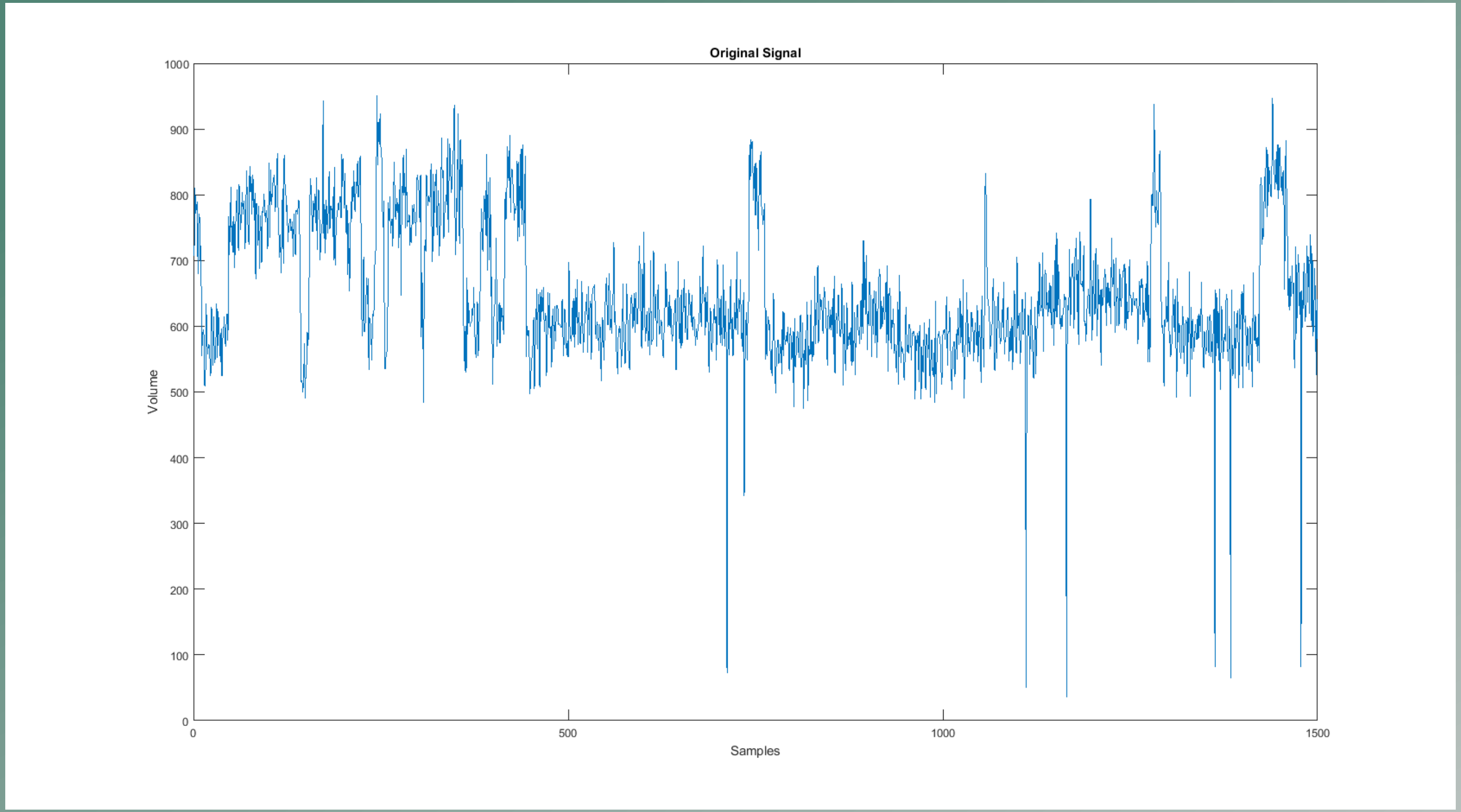
Wavelet Denoising

- Once details coefficients below the threshold values are set to zero, the inverse wavelet transform is performed to bring the new signal to the same size as the raw signal (No Compression)

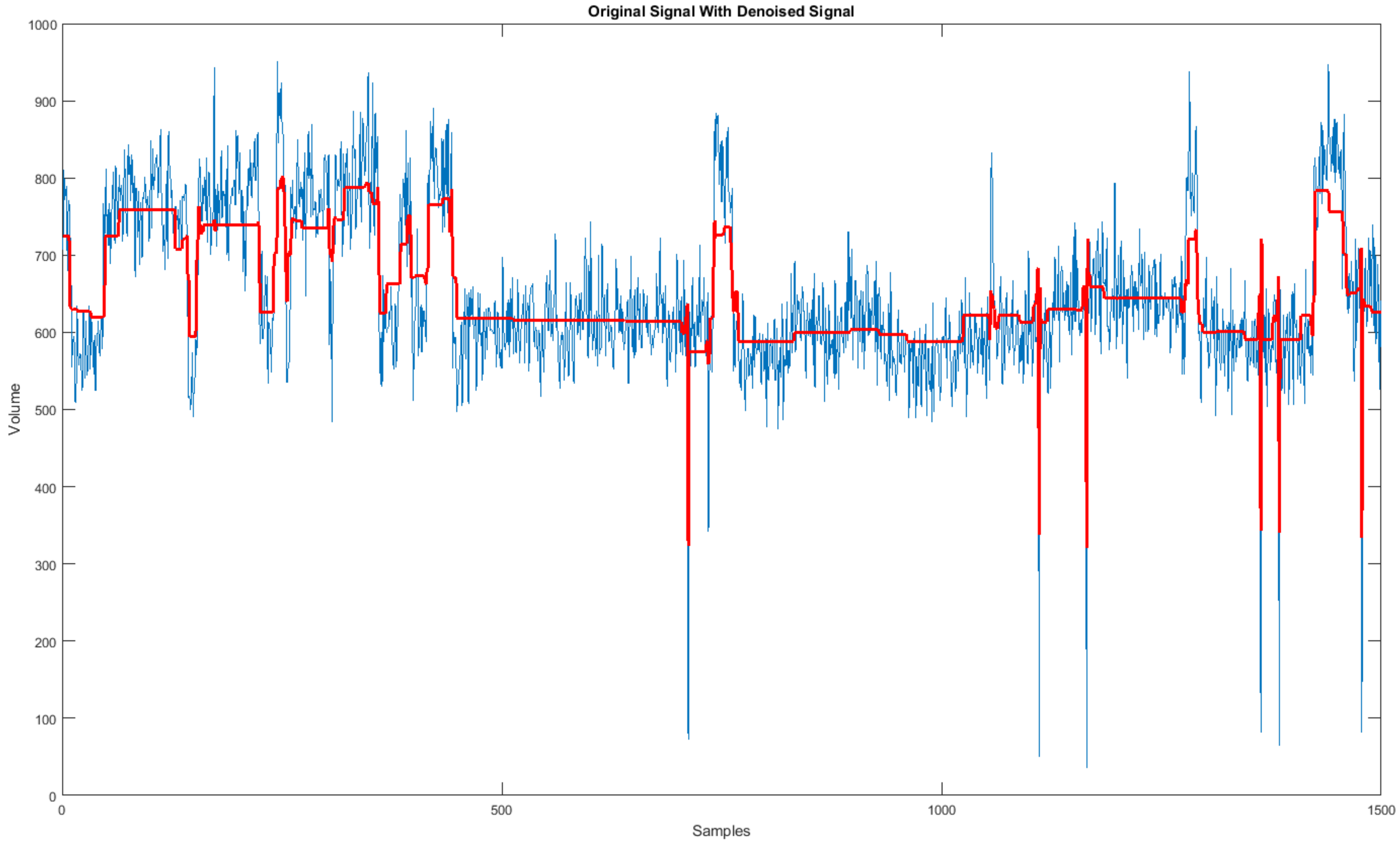
$$\mathbf{f} = \left(\frac{a_1 + d_1}{\sqrt{2}}, \frac{a_1 - d_1}{\sqrt{2}}, \ldots, \frac{a_{\frac{N}{2}} + d_{\frac{N}{2}}}{\sqrt{2}}, \frac{a_{\frac{N}{2}} - d_{\frac{N}{2}}}{\sqrt{2}}\right)$$

Signal construction – Raw Signal

- The goal is to take a noisy signal, and construct an accurate approximation with zero erroneous noise

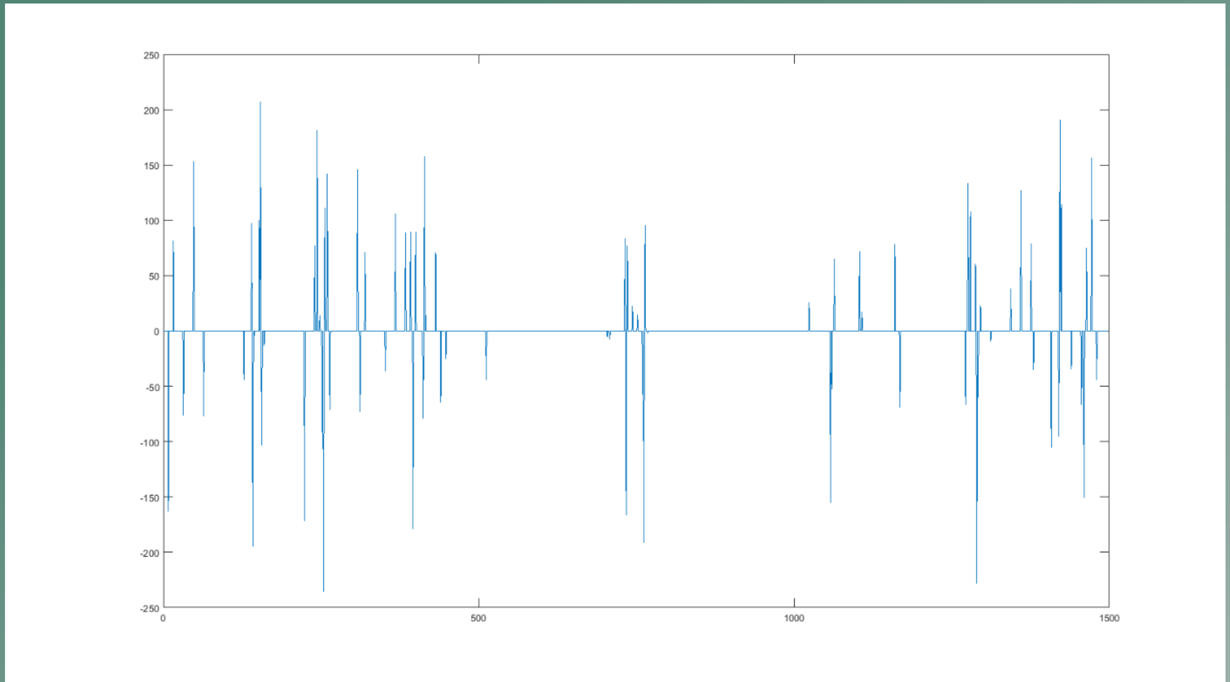


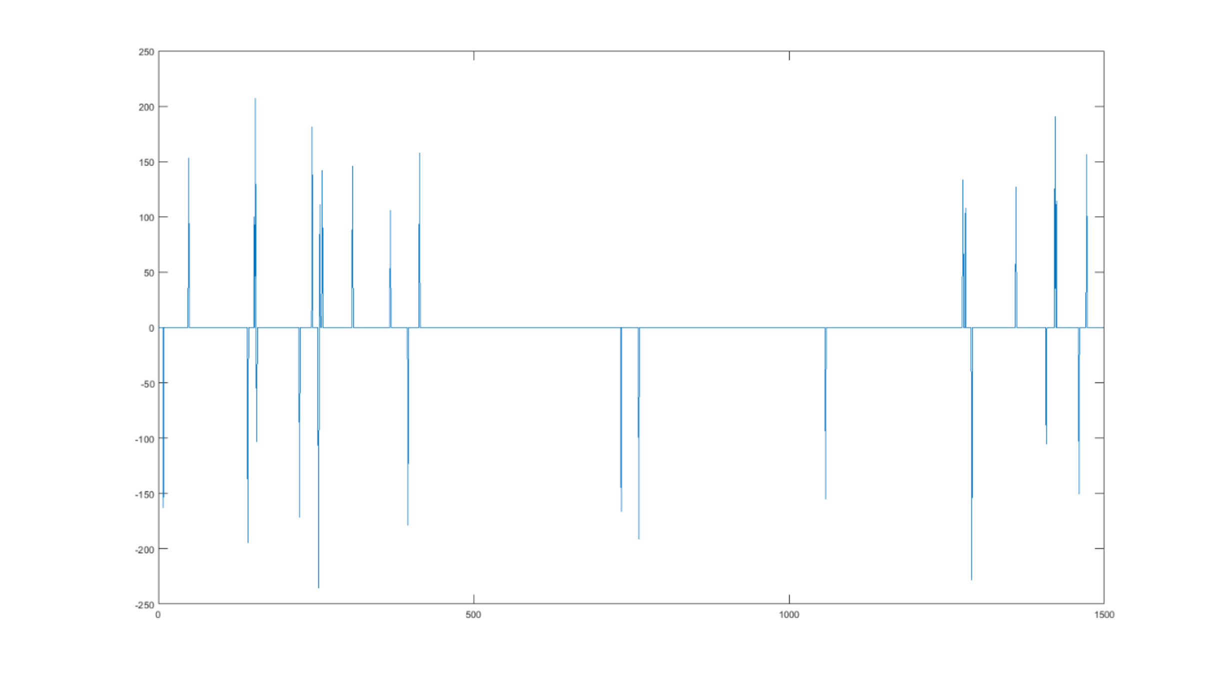
Signal Construction – Wavelet Denoised



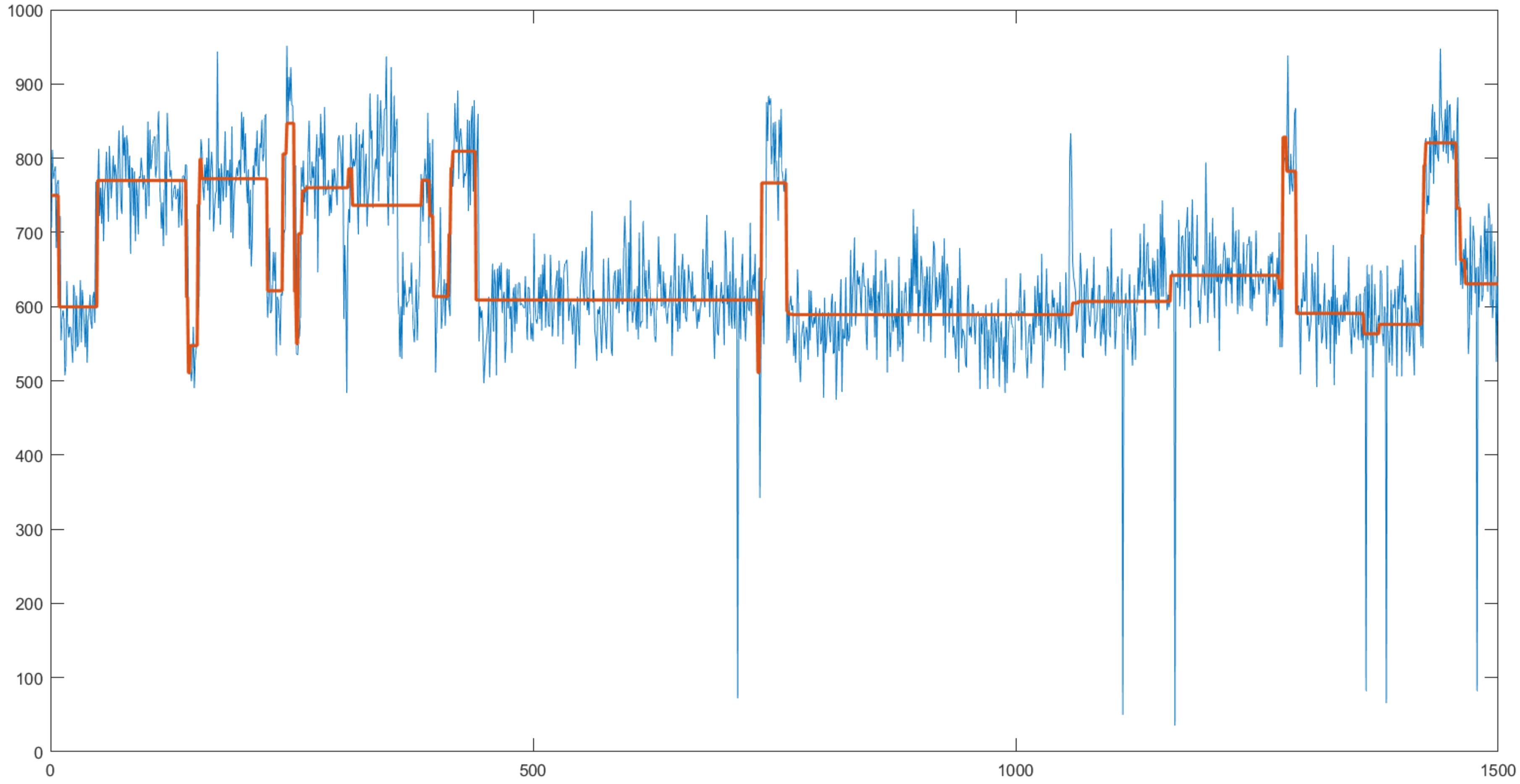
Arithmetic Subtraction

- A new ‘details’ signal to threshold
- Non-Dyadic – Returns a signal of length N-1
- Threshold no longer needed to be as discriminatory
- Must remove single events





Signal Construction – Mean between Peaks



Signal Construction – Discussion

- Where it succeeds
  - On a good detect, it very accurately captures RTS time constants and amplitudes
    Does a good job deleting single events

- Room for improvement
  - Still misses some transitions.